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Research Article

New Algorithm Based on Sign Decomposition to Verify the Robust Stability Property for a Class of Linear Time-Delay Systems

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The main focus of this paper is to analyze the robust stability property for a class of time-delay systems when parametric polynomic uncertainty is considered. The analysis is made by replacing the time-delay part with an auxiliary equation and then using the sign definite decomposition to deal with the polynomic parametric uncertainty. Also, it is shown that it is possible to verify the robust stability property by first obtaining the Hurwitz matrix from the characteristic equation for this class of systems and then checking the leading principal minors positivity using the sign definite decomposition. Finally, an algorithm codified in MATLAB is used to evaluate and graphically show the robust stability property. This is shown by a series of points that were calculated using the sign definite decomposition.

1. Introduction

Time-delay systems arose as a result of inherent delays in system's components and also due to the introduction of deliberated delay in the system for control purposes; see [1–3]. Over the years, time-delay systems interest and popularity have grown steadily. In particular, in the last 10 to 15 years there has been a surge in research and a proliferation of new techniques and results. Many of these new results include systems not only with time-delay analysis, but also with uncertainty in the system to be considered. For example, in [4–6] an analysis of robust stability for time-delay dynamical systems with parametric uncertainty in the mathematical model and in the time-delay is shown; in addition, the value set concept and the zero exclusion condition are used to verify the robust stability property of interval plants; see also [7, 8]. In [9, 10] the robust stabilization problem for a class of time-delay systems is considered where they involve parametric affine perturbations; in [11], new results to compute the time-delay of the hot-dip galvanizing control system are considered. In [12, 13] the authors present a robust model predictive control for systems represented by Takagi-Sugeno models and this technique was applied to the continuous stirred tank reactor (CSTR). They use Linear Matrix Inequalities (LMI) to solve the optimization problem. A technique based on a representation in the time domain of a class of *differential-difference* systems is presented in [14, 15]. Here, the authors make an application to the active suspension systems with actuator delay using the aforementioned technique. In [16, 17] new robust stability results for LTI systems with parametric uncertainty using sign definite decomposition were developed. In [18] the robust stability problem for a polynomial family was considered whose coefficients are polynomial functions of the parameters of interest. They used the sign definite decomposition for the controllers design. In [19] the robust stability positivity of a real function $f(x)$ is considered while the real vector x varies over a box. They determined Hurwitz robust stability for a polynomial family using the sign definite decomposition described in [16].

In 1981, the characteristic polynomial including a time-delay, for a linear differential-difference system, considered replacing the term $e^{-s\tau}$ by a regular polynomial $(1-Ts)^2/(1+Ts)^2$. After this, it was possible to verify the asymptotic stability property for a class of time-delay systems; see [20]. Previously, a different approach was made in [21] by replacing $e^{-s\tau}$ by $(1-Ts)/(1+Ts)$. But, it was found in [20] that the main problem with this substitution was that the two sets of image points were not identical for all $s = j\omega$, $\omega > 0$, $T > 0$ and $\tau > 0$. In the present paper we outline a new algorithm to verify the robust stability property for a class of linear time-delay systems including a special case of polynomic parametric uncertainty, which one has not been considered in systems involving a time-delay. This is by using sign definite decomposition theory to verify the robust stability of the system using the stability conditions of Hurwitz matrix. We illustrate this using a numerical example.

This paper is organized as follows. In the preliminaries section, the Hurwitz stability criterion, a special polynomic parametric uncertainty case, and the sign definite decomposition are described. Then, the problem statement is presented. After that, the methodology and proposed algorithm are shown. An illustrative numerical example is presented to show the effectiveness of this approach. Finally, we discuss our results and future research.

2. Main Contribution

As it may be seen from the section above, some of the previous results use techniques based on a representation in the time domain of *differential-difference* systems. Thus, to analyze and design them it is necessary to use the Lyapunov technique. It should be also mentioned that the uncertainty that they experiment should be represented by time functions. However, there are many applications where the uncertainty depends on variables other than time, such as resistors, capacitors, and inductors in an electrical circuit, which have parameters that are uncertain and whose uncertainty depends mainly on temperature and therefore could not be analyzed with these techniques. Also, in previous section some other results were mentioned that consider uncertainty structures like interval or linear affine and systems without delay. However, the main result of this paper is to obtain sufficient conditions to verify the robust stability property of a class of quasi-polynomials that represent the characteristic equation of *differential-difference* dynamics systems. It considers polynomic parametric uncertainty structure in the coefficients of quasi-polynomials and also interval uncertainty in the time-delay. First of all, a transformation of the delay's operator is performed in order to get a two-variable polynomial; after this, to obtain the robust stability property, a result based on the Hurwitz matrix is applied, and then checking the leading principal minors positivity using the sign definite decomposition.

3. Preliminaries

3.1. Hurwitz Stability Criterion

Theorem 1 (Hurwitz stabilty). *Given a real polynomial $p(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$, the polynomial $p(s)$ is stable; that is, all its roots lie in the open left half plane (LHP) of the complex plane, if and only if, all of the leading principal minors, defined by Δ_i, of the matrix H are positive; see [22]:*

$$H = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ 0 & a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}. \quad (1)$$

3.2. Uncertainty. There exists a case where the precise value of the parameters of the mathematical model is unknown; however, its lower and upper bounds are known q_i^-, q_i^+, respectively. The collection of all l parameters involved in the mathematical model forms a vector of parameters $q = [q_1, q_2, \ldots, q_l]^T$ which is an element of a parametric uncertainty box Q:

$$Q = \{q = [q_1, q_2, \ldots, q_l]^T \mid q_i \in [q_i^-, q_i^+],\ i = 1, 2, \ldots, l\}. \quad (2)$$

For different lower and upper bound values, it is always possible to make a coordinate transformation of the physical parameters without losing their original properties. Such transformation can be $\rho_i = [q_i - q_i^-]/[q_i^+ - q_i^-]$, and in this case $q_i \in [q_i^-, q_i^+]$ is taken in $\rho_i = [0, 1]$, where, for simplicity, we can name q to the new coordinate $q_i = [0, 1]$. When we consider a parametric uncertainty, we have a polynomial family defined as

$$P(s, Q) \triangleq \{p(s, q) : q \in Q\}. \quad (3)$$

There exist a class of polynomials with parametric uncertainty $p(s, q)$. It is called *polynomic uncertainty structure*; that is, it has all of its coefficients where at least one parameter appears with power greater than one. For example, $p(s, q) = (q_1 + 2q_1^2 q_3)s^2 + (q_1 q_2^3 + q_1)s + (2q_2 q_3)$.

Definition 2 (see [23]). Let P be a positive convex cone in a vector space $\mathbb{R}^l$, for all $x, y \in \mathbb{R}^l$, it is said that $x \geq y$ $(x > y)$ with respect to P if $x - y \in P$ $(x - y \in P^0$, the interior of $P)$.

From this point, we will consider $Q \subset P$ and $q_i^- \geq 0$. This implies that $q \in Q \subset P$.

Definition 3 (see [23]). $f : \mathbb{R}^l \to \mathbb{R}$ being a continuous function and $Q \subset P \subset \mathbb{R}^l$ a convex subset, it is said that $f(\cdot)$ is a nondecreasing function in Q, if $x \geq y$ implies $f(x) \geq f(y)$, $\forall x, y \in Q$.

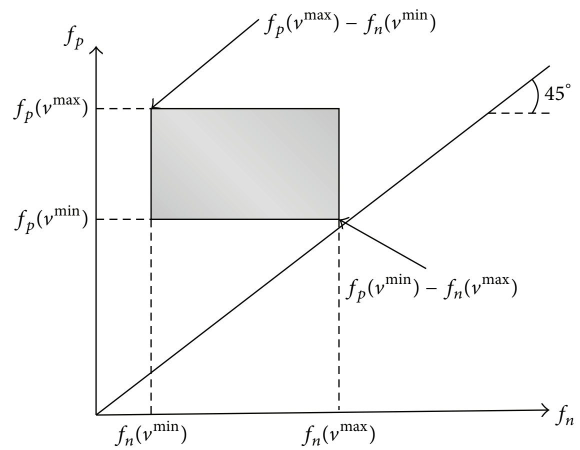

Figure 1: Rectangle containing the function $f(q)$.

3.3. Sign Definite Decomposition

Definition 4 (see [16, 17]). $f : \mathbb{R}^l \to \mathbb{R}$ being a continuous function and $Q \subset P \subset \mathbb{R}^l$ a convex subset, it is said that $f(\cdot)$ has sign decomposition in Q if there exist two nondecreasing bounded functions $f_n(\cdot) \geq 0$, $f_p(\cdot) \geq 0$, such that $f(q) = f_p(q) - f_n(q)$ for all $q \in Q$. One will call those functions the positive $f_p(\cdot)$ and negative $f_n(\cdot)$ parts of the function:

$$\begin{gathered} f(q) = f_p(q) - f_n(q) \quad \forall q \in Q \\ f_p(\cdot) \triangleq \text{Positive part of } f(\cdot) \\ f_n(\cdot) \triangleq \text{Negative part of } f(\cdot). \end{gathered} \tag{4}$$

The negative and positive parts $(f_n(\cdot), f_p(\cdot))$ constitute a representation (f_n, f_p) of the function in $\mathbb{R}^2$ with a graphic representation in the plane $(f_n(\cdot), f_p(\cdot))$ according to Figure 1.

Definition 5 (see [16, 17]). It will be called minimum and maximum euclidean vertex $v^{\min}$, $v^{\max}$ to the vectors elements of $Q \subset P \subset \mathbb{R}^\ell$ with the minimum and maximum Euclidean norm, respectively:

$$\begin{gathered} \left\|v^{\min}\right\|_2 = \min_{q\in Q} \|q\|_2, \\ \left\|v^{\max}\right\|_2 = \max_{q\in Q} \|q\|_2. \end{gathered} \tag{5}$$

Since the negative $f_n(q)$ and positive $f_p(q)$ parts are nondecreasing functions in a vector space, the graphic representation of $f(q)$, $\forall q \in Q$ in the plane (f_n, f_p) is contained in Figure 1, where if the lower right vertex $(f_n(v^{\max}), f_p(v^{\min}))$ is above the 45° line, then the function $f(q) > 0$, $\forall q \in Q$.

Definition 6 (see [16, 17]). $f_p(q)$ and $f_n(q)$ are the elements of a function $f(q)$ with sign definite decomposition in Q. T being the linear transformation described such that there exists T^{-1}, then it is called a representation of the function $f(q)$ in (α, β) coordinates to the linear transformation $(\alpha(q), \beta(q)) = T(f_n(q), f_p(q))$ of the function:

$$\begin{gathered} T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ T^{-1} = 0.5 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} \alpha(q) \\ \beta(q) \end{bmatrix} = T \begin{bmatrix} f_n(q) \\ f_p(q) \end{bmatrix} \\ \begin{bmatrix} f_n(q) \\ f_p(q) \end{bmatrix} = T^{-1} \begin{bmatrix} \alpha(q) \\ \beta(q) \end{bmatrix} \\ \alpha(q) = f_p(q) + f_n(q) \\ f_p(q) = 0.5(\alpha(q) + \beta(q)) \\ \beta(q) = f_p(q) - f_n(q) \\ f_n(q) = 0.5(\alpha(q) - \beta(q)). \end{gathered} \tag{6}$$

In order to define the positivity or negativity of a function using the (α, β) representation, when a polynomial uncertainty set is included, we need to use the following theorem.

Theorem 7 ((rectangle) [16, 17]). *$f : \mathbb{R}^l \to \mathbb{R}$ being a continuous function with sign definite decomposition in Q such that $Q \subset P \subset \mathbb{R}^l$ is a box with euclidean minimum and maximum vertex $v^{\min}$, $v^{\max}$ then (a) the lower bound of $f(q)$ is $0.5[\alpha(v^{\min}) + \beta(v^{\min}) - \alpha(v^{\max}) + \beta(v^{\max})]$ and its upper bound is $0.5[\alpha(v^{\max}) + \beta(v^{\max}) - \alpha(v^{\min}) + \beta(v^{\min})]$, respectively; (b) the graphic representation of the function $f(q)$, $\forall q \in Q$ in the plane (α, β) is contained in the rectangle with vertices: $\alpha^{izq} = \alpha(v^{\min})$, $\beta^{izq} = \beta(v^{\min})$, $\alpha^{\text{der}} = \alpha(v^{\max})$, $\beta^{\text{der}} = \alpha(v^{\max})$; $\alpha^{\inf} = 0.5[\alpha(v^{\min}) + \alpha(v^{\max})] - 0.5[\beta(v^{\max}) - \beta(v^{\min})]$, $\beta^{\inf} = 0.5[\beta(v^{\min}) + \beta(v^{\max})] - 0.5[\alpha(v^{\max}) - \alpha(v^{\min})]$, $\alpha^{\sup} = 0.5[\alpha(v^{\min}) + \alpha(v^{\max})] + 0.5[\beta(v^{\max}) - \beta(v^{\min})]$, $\beta^{\sup} = 0.5[\beta(v^{\min}) + \beta(v^{\max})] + 0.5[\alpha(v^{\max}) - \alpha(v^{\min})]$; (c) if the lower vertex $(\alpha^{\inf}, \beta^{\inf})$ is above the α axis in the (α, β) plane, then the function $f(q) < 0$, $\forall q \in Q$.*

Theorem 8 ((box partition) [16, 17]). *$f : \mathbb{R}^l \to \mathbb{R}$ being a continuous function with sign definite decomposition in Q such that $Q \subset P \subset \mathbb{R}^l$ is a box with minimum and maximum euclidean vertices $v^{\min}, v^{\max}$ then the function is positive (negative) in Q if and only if there exist a set of boxes Γ, such that $Q = \bigcup_j \Gamma^j$ and the lower bound $\geq c > 0$ for every box Γ^j (upper bound $\leq c < 0$ for every box Γ^j).*

The determinant of the matrix M is comprised of additions and subtractions of products of the elements of the matrix and if those are formed of polynomial type, the determinant $\det(M)$ has sign definite decomposition. The programming development in order to get the sign definite decomposition of the determinant in the representation

(f_n, f_p) can be quite complicated; however, in the (α, β) representation there exists a less complicated way to do it.

Definition 9 (see [16, 17]). $M(q)$ being a square matrix with elements $m_{i,j}(q)$ with sign definite decomposition in Q with representation $(\alpha_{i,j}(q), \beta_{i,j}(q))$, then it will be called $M_\alpha(q)$ to the matrix formed with the elements $\alpha_{i,j}(q)$ and it will be called $\det_\alpha(M_\alpha(q)) = |M_\alpha(q)|_\alpha$ to the function similar to the determinant of the matrix $M_\alpha(q)$ but without applying the sign rule $(-1)^{i+j}$; it will be $M_\beta(q) = M(q)$ and $\det_\beta(M_\beta(q)) = \det(M(q))$.

Lemma 10 (see [16, 17]). *Let $M(q)$ be a square matrix with elements $m_{i,j}(q)$ with sign definite decomposition in Q with $(\alpha_{i,j}(q), \beta_{i,j}(q))$ representation. $M_\alpha(q)$ being the square matrix with $\alpha_{i,j}(q)$ elements, then the (α, β) representation of the matrix determinant $M(q)$ is given by*

$$\begin{aligned} \alpha(q) &= \det_\alpha(M_\alpha(q)), \\ \beta(q) &= \det(M(q)). \end{aligned} \tag{7}$$

4. Problem Statement

The main interest of this research is to analyze the robust stability property of difference-differential dynamical systems which are characterized by polynomic parametric uncertainty and time-delay of the form:

$$\dot{x}(t) = A_0(q)\,x(t) + A_1(q)\,x(t-\tau), \tag{8}$$

where $A_0(q), A_1(q) \in \mathbb{R}^{nxn}$ are matrices with dependent parameters of $q_i \in Q$ and $\tau \in [0, \tau_{\max}]$; for example,

$$\begin{aligned} A_0(q) &= \begin{bmatrix} q_1^2 q_2 & q_3^4 \\ q_1 q_2 & q_1^2 q_2^2 q_3^2 \end{bmatrix} \\ A_1(q) &= \begin{bmatrix} q_3^2 & q_1^2 q_2^3 q_3 \\ q_1^5 q_2 & q_3 \end{bmatrix}, \\ q &= \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad q_i \in [q_i^-, q_i^+]. \end{aligned} \tag{9}$$

The parameters q_i represent the polynomic structure uncertainty and τ the uncertain time-delay. Then, system (8) is asymptotically stable if and only if the following condition is satisfied:

$$p(s, q, e^{-\tau s}) = \det\{sI - A_0(q) - A_1(q)\,e^{-\tau s}\} \neq 0. \tag{10}$$

One has $\forall s \in \mathbb{C}_+$, wWhere $\mathbb{C}_+$ represents the RHP of the complex plane. The quasi-polynomials that satisfy the last condition are called stable quasi-polynomials.

5. Results

The robust stability property is determined by the analysis of the characteristic equation (10). Such equation is called characteristic quasi-polynomial with polynomic parametric uncertainty. The following transformation is needed in order to determine the robust stability condition for this class of systems.

Definition 11. A polynomial $p(s, T)$ associated with a quasi-polynomial $p(s, e^{-\tau s})$ will be defined as follows:

$$\begin{aligned} p(s, T) &= \sum_{i=0}^{n} p_i(s)\,(1 - Ts)^{2i}\,(1 + Ts)^{2n-2i} \\ p(s, e^{-\tau s}) &= \sum_{i=0}^{n} p_i(s)\,e^{-i\tau s}. \end{aligned} \tag{11}$$

The roots of this associated polynomial have an important relation with the roots of the quasi-polynomial. This relation is presented in the following theorem.

Theorem 12 (see [20]). *Suppose that $s^0 = j\omega^0$ for some value of $\omega^0 \geq 0$; then $s^0 = j\omega^0$ is a matrix of the characteristic equation $p(s, e^{-\tau s})$ for some value of $\tau \geq 0$ if and only if $s^0 = j\omega^0$ is a root of $p(s, T)$ for some value of $T \geq 0$.*

With this transformation we can get the relation between τ and T, which is valid in the imaginary axis $j\omega$; see [20]. The time-delay τ and T are related by the following equation:

$$T = \frac{\tan(\tau\omega_i/4)}{\omega_i}, \tag{12}$$

where $\omega_i \in W$; the set of W is defined as follows:

$$W \equiv \{0 < \omega < \omega^* : p(j\omega^*, T) = 0 \text{ para } T > 0\}, \tag{13}$$

where $p(j\omega, T)$ is a polynomial associated with $p(s, e^{-\tau s})$ evaluated in the $j\omega$ axis. Note that for each value of ω_i, there exists a direct relation τ and T; also, this relation is a continuous function and strictly increasing in the range $T \in [0, \infty)$. For this reason, for every fixed value of ω_i, the interval $\tau \in [0, \tau_{\max}]$ generates an interval $T \in [0, T_{\max}]$. Now, it is clear that for all values of $\omega_i \in W$ there exists a relation between τ and T; we will define $T_{\max}$ in the following relation:

$$T_{\max} = \min\{T_i\} \quad T_i = \frac{\tan(\tau\omega_i/4)}{\omega_i} \ \forall \omega_i \in W. \tag{14}$$

Definition 13. The Hurwitz matrix is

$$\begin{aligned} &H[p(s, q, T)] \\ &= \begin{bmatrix} h_{1,1}(q,T) & h_{1,2}(q,T) & \cdots & 0 \\ h_{2,1}(q,T) & h_{2,2}(q,T) & \cdots & 0 \\ 0 & h_{3,2}(q,T) & \cdots & 0 \\ 0 & h_{4,2}(q,T) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & h_{n,n}(q,T) \end{bmatrix}, \end{aligned} \tag{15}$$

where every element of the matrix depends of the values of $q_i \in Q \subset P$ and $T \in (0, T_{\max}]$.

Definition 14. The Hurwitz matrix being (15), then $f_{p_{i,j}}(q,T)$ and $f_{n_{i,j}}(q,T)$ will be denoted to the positive and negative parts, respectively, for every element of the Hurwitz matrix such that

$$h_{i,j}(q,T) = f_{p_{i,j}}(q,T) + f_{n_{i,j}}(q,T), \tag{16}$$

where $i = 1,2,\dots,n$, $j = 1,2,\dots,n$.

According to the (α,β) representation, we can express the following.

Definition 15. Let $\Delta_i[\alpha^{\inf}]$, $i = 1,2,\dots,n$ be the matrices with elements $\alpha_{i,j}^{\inf}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\begin{gathered}\Delta_1\left[\alpha^{\inf}\right] = \left[\alpha_{1,1}^{\inf}\right]\\ \Delta_2\left[\alpha^{\inf}\right] = \begin{bmatrix}\alpha_{1,1}^{\inf} & \alpha_{1,2}^{\inf}\\ \alpha_{2,1}^{\inf} & \alpha_{2,2}^{\inf}\end{bmatrix}\\ \vdots\\ \Delta_n\left[\alpha^{\inf}\right] = \begin{bmatrix}\alpha_{1,1}^{\inf} & \alpha_{1,2}^{\inf} & \alpha_{1,3}^{\inf} & 0 & \cdots & 0\\ \alpha_{2,1}^{\inf} & \alpha_{2,2}^{\inf} & \alpha_{2,3}^{\inf} & 0 & \cdots & 0\\ 0 & \alpha_{3,2}^{\inf} & \alpha_{3,3}^{\inf} & \alpha_{3,4}^{\inf} & \cdots & 0\\ 0 & \alpha_{4,2}^{\inf} & \alpha_{4,3}^{\inf} & \alpha_{4,4}^{\inf} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n,n}^{\inf}\end{bmatrix},\end{gathered} \tag{17}$$

where every element of the matrices $\alpha_{i,j}^{\inf} = f_p(\nu^{\min}) + f_n(\nu^{\max})$ is formed by the addition of the positive and negative parts taken from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$.

Definition 16. Let $\Delta_i[\beta^{\inf}]$, $i = 1,2,\dots,n$, be the matrices with elements $\beta_{i,j}^{\inf}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\begin{gathered}\Delta_1\left[\beta^{\inf}\right] = \left[\beta_{1,1}^{\inf}\right]\\ \Delta_2\left[\beta^{\inf}\right] = \begin{bmatrix}\beta_{1,1}^{\inf} & \beta_{1,2}^{\inf}\\ \beta_{2,1}^{\inf} & \beta_{2,2}^{\inf}\end{bmatrix}\\ \vdots\\ \Delta_n\left[\beta^{\inf}\right] = \begin{bmatrix}\beta_{1,1}^{\inf} & \beta_{1,2}^{\inf} & \beta_{1,3}^{\inf} & 0 & \cdots & 0\\ \beta_{2,1}^{\inf} & \beta_{2,2}^{\inf} & \beta_{2,3}^{\inf} & 0 & \cdots & 0\\ 0 & \beta_{3,2}^{\inf} & \beta_{3,3}^{\inf} & \beta_{3,4}^{\inf} & \cdots & 0\\ 0 & \beta_{4,2}^{\inf} & \beta_{4,3}^{\inf} & \beta_{4,4}^{\inf} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \beta_{n,n}^{\inf}\end{bmatrix},\end{gathered} \tag{18}$$

where every element of the matrices $\beta_{i,j}^{\inf} = f_p(\nu^{\min}) - f_n(\nu^{\max})$ is formed by the subtraction of the positive and negative parts taken from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$.

Definition 17. Let $\Delta_i[\alpha^{\mathrm{der}}]$, $i = 1,2,\dots,n$, be the matrices with elements $\alpha_{i,j}^{\mathrm{der}}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\begin{gathered}\Delta_1\left[\alpha^{\mathrm{der}}\right] = \left[\alpha_{1,1}^{\mathrm{der}}\right]\\ \Delta_2\left[\alpha^{\mathrm{der}}\right] = \begin{bmatrix}\alpha_{1,1}^{\mathrm{der}} & \alpha_{1,2}^{\mathrm{der}}\\ \alpha_{2,1}^{\mathrm{der}} & \alpha_{2,2}^{\mathrm{der}}\end{bmatrix}\\ \vdots\\ \Delta_n\left[\alpha^{\mathrm{der}}\right] = \begin{bmatrix}\alpha_{1,1}^{\mathrm{der}} & \alpha_{1,2}^{\mathrm{der}} & \alpha_{1,3}^{\mathrm{der}} & 0 & \cdots & 0\\ \alpha_{2,1}^{\mathrm{der}} & \alpha_{2,2}^{\mathrm{der}} & \alpha_{2,3}^{\mathrm{der}} & 0 & \cdots & 0\\ 0 & \alpha_{3,2}^{\mathrm{der}} & \alpha_{3,3}^{\mathrm{der}} & \alpha_{3,4}^{\mathrm{der}} & \cdots & 0\\ 0 & \alpha_{4,2}^{\mathrm{der}} & \alpha_{4,3}^{\mathrm{der}} & \alpha_{4,4}^{\mathrm{der}} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n,n}^{\mathrm{der}}\end{bmatrix},\end{gathered} \tag{19}$$

where every element of the matrices $\alpha_{i,j}^{\mathrm{der}} = f_p(\nu^{\max}) + f_n(\nu^{\max})$ is formed by the addition of the positive and negative parts from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$.

Definition 18. Let $\Delta_i[\beta^{\mathrm{der}}]$, $i = 1,2,\dots,n$, be the matrices with elements $\beta_{i,j}^{\mathrm{der}}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\begin{gathered}\Delta_1\left[\beta^{\mathrm{der}}\right] = \left[\beta_{1,1}^{\mathrm{der}}\right]\\ \Delta_2\left[\beta^{\mathrm{der}}\right] = \begin{bmatrix}\beta_{1,1}^{\mathrm{der}} & \beta_{1,2}^{\mathrm{der}}\\ \beta_{2,1}^{\mathrm{der}} & \beta_{2,2}^{\mathrm{der}}\end{bmatrix}\\ \vdots\end{gathered}$$

$$\Delta_n\left[\beta^{\text{der}}\right]=\begin{bmatrix}\beta_{1,1}^{\text{der}} & \beta_{1,2}^{\text{der}} & \beta_{1,3}^{\text{der}} & 0 & \cdots & 0\\ \beta_{2,1}^{\text{der}} & \beta_{2,2}^{\text{der}} & \beta_{2,3}^{\text{der}} & 0 & \cdots & 0\\ 0 & \beta_{3,2}^{\text{der}} & \beta_{3,3}^{\text{der}} & \beta_{3,4}^{\text{der}} & \cdots & 0\\ 0 & \beta_{4,2}^{\text{der}} & \beta_{4,3}^{\text{der}} & \beta_{4,4}^{\text{der}} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \beta_{n,n}^{\text{der}}\end{bmatrix}, \tag{20}$$

where every element of the matrices $\beta_{i,j}^{\text{der}} = f_p(v^{\max}) - f_n(v^{\max})$ is formed by the subtraction of the positive and negative parts taken from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$.

Definition 19. Let $\Delta_i[\alpha^{izq}]$, $i = 1,2,\ldots,n$, be the matrices with elements $\alpha_{i,j}^{izq}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\Delta_1\left[\alpha^{izq}\right]=\left[\alpha_{1,1}^{izq}\right]$$

$$\Delta_2\left[\alpha^{izq}\right]=\begin{bmatrix}\alpha_{1,1}^{izq} & \alpha_{1,2}^{izq}\\ \alpha_{2,1}^{izq} & \alpha_{2,2}^{izq}\end{bmatrix}$$

$$\vdots$$

$$\Delta_n\left[\alpha^{izq}\right]=\begin{bmatrix}\alpha_{1,1}^{izq} & \alpha_{1,2}^{izq} & \alpha_{1,3}^{izq} & 0 & \cdots & 0\\ \alpha_{2,1}^{izq} & \alpha_{2,2}^{izq} & \alpha_{2,3}^{izq} & 0 & \cdots & 0\\ 0 & \alpha_{3,2}^{izq} & \alpha_{3,3}^{izq} & \alpha_{3,4}^{izq} & \cdots & 0\\ 0 & \alpha_{4,2}^{izq} & \alpha_{4,3}^{izq} & \alpha_{4,4}^{izq} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n,n}^{izq}\end{bmatrix}, \tag{21}$$

where every element of the matrices $\alpha_{i,j}^{izq} = f_p(v^{\min}) + f_n(v^{\min})$ is formed by the addition of the positive and negative parts from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$.

Definition 20. Let $\Delta_i[\beta^{izq}]$, $i = 1,2,\ldots,n$, be the matrices with elements $\beta_{i,j}^{izq}$ obtained from the leading principal minors Δ_i of the Hurwitz matrix $H[p(s,q,T)]$:

$$\Delta_1\left[\beta^{izq}\right]=\left[\beta_{1,1}^{izq}\right]$$

$$\Delta_2\left[\beta^{izq}\right]=\begin{bmatrix}\beta_{1,1}^{izq} & \beta_{1,2}^{izq}\\ \beta_{2,1}^{izq} & \beta_{2,2}^{izq}\end{bmatrix}$$

$$\vdots$$

$$\Delta_n\left[\beta^{izq}\right]=\begin{bmatrix}\beta_{1,1}^{izq} & \beta_{1,2}^{izq} & \beta_{1,3}^{izq} & 0 & \cdots & 0\\ \beta_{2,1}^{izq} & \beta_{2,2}^{izq} & \beta_{2,3}^{izq} & 0 & \cdots & 0\\ 0 & \beta_{3,2}^{izq} & \beta_{3,3}^{izq} & \beta_{3,4}^{izq} & \cdots & 0\\ 0 & \beta_{4,2}^{izq} & \beta_{4,3}^{izq} & \beta_{4,4}^{izq} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \cdots & \beta_{n,n}^{izq}\end{bmatrix}, \tag{22}$$

where every element of the matrices $\beta_{i,j}^{izq} = f_p(v^{\min}) - f_n(v^{\min})$ is formed by the subtraction of the positive and negative parts from the corresponding element of the Hurwitz matrix $H[p(s,q,T)]$. Now, the main result is presented. This presents the robust stability condition for the class of systems described in (1).

5.1. Robust Stability Analysis Algorithm. In this subsection we describe the steps to follow to analyze the robust stability property of the system including time-delay and polynomic parametric uncertainty. The algorithm is the following.

Step 1. Consider the time-delay system described in (8) and the characteristic equation in (10).

Step 2. Determine the characteristic equation $p(s,q,T)$ using (11).

Step 3. Define the Hurwitz matrix $H[p(s,q,T)]$ given by (15).

Step 4. For each element in the Hurwitz matrix, perform a separation in positive and negative part. This is given in (16).

Step 5. With respect to the (α,β) representation given in the preliminaries, define the leading principal minors from the (α,β) representation taking the Hurwitz matrix $H[p(s,q,T)]$. That is, $\Delta_i[\alpha_i^{\text{inf}}]$, $\Delta_i[\beta_i^{\text{inf}}]$, $\Delta_i[\alpha_i^{\text{der}}]$, $\Delta_i[\beta_i^{\text{der}}]$, $\Delta_i[\alpha_i^{izq}]$ and $\Delta_i[\beta_i^{izq}]$ given in (17), (18), (19), (20), (21), and (22), respectively.

Step 6. Determine the points $(\alpha_i^{\text{inf}},\beta_i^{\text{inf}})$, $(\alpha_i^{\text{der}},\beta_i^{\text{der}})$, and $(\alpha_i^{izq},\beta_i^{izq})$ in the (α,β) plane obtained by $\alpha_i^{\text{inf}} = \det(\Delta_i[\alpha_i^{\text{inf}}])$, $\beta_i^{\text{inf}} = \det(\Delta_i[\beta_i^{\text{inf}}])$, $\alpha_i^{\text{der}} = \det(\Delta_i[\alpha_i^{\text{der}}])$, $\beta_i^{\text{der}} = \det(\Delta_i[\beta_i^{\text{der}}])$, $\alpha_i^{izq} = \det(\Delta_i[\alpha_i^{izq}])$, and $\beta_i^{izq} = \det(\Delta_i[\beta_i^{izq}])$.

Step 7. Verify the following:

(i) If the points $(\alpha_i^{\text{inf}},\beta_i^{\text{inf}})$, $(\alpha_i^{\text{der}},\beta_i^{\text{der}})$, and $(\alpha_i^{izq},\beta_i^{izq})$ are located above of the α axis, then the system described in (8) is robustly stable. Stop the algorithm.

(ii) If the points $(\alpha_i^{\text{inf}},\beta_i^{\text{inf}})$, $(\alpha_i^{\text{der}},\beta_i^{\text{der}})$, and $(\alpha_i^{izq},\beta_i^{izq})$ are located below the α axis, then the system described in (8) is not robustly stable. Stop the algorithm.

(iii) If we have at least one point $(\alpha_i^{\text{der}},\beta_i^{\text{der}})$ or $(\alpha_i^{izq},\beta_i^{izq})$ located below the α axis, the analysis is concluded with the knowledge that the system described in (8) is not robustly stable. Stop the algorithm.

(iv) If the points $(\alpha_i^{izq}, \beta_i^{izq})$, $(\alpha_i^{\text{der}}, \beta_i^{\text{der}})$ are located above the α axis but the points $(\alpha_i^{\text{inf}}, \beta_i^{\text{inf}})$ are located below the α axis, then we can make a partition in a subset of boxes Γ^i as given in Theorem 8. Go to Step 5.

6. Example

Consider a first order system:

$$\dot{x}(t) = -x(t) - 2x(t-\tau), \quad \tau > 0. \tag{23}$$

Using Definition 11, we have that the system has the following auxiliary equation:

$$s + 1 + 2\left(\frac{1-Ts}{1+Ts}\right)^2 = 0. \tag{24}$$

For this polynomial, we can find out that it has roots in the LHP if and only if $T = 1/3$ and they are located in $s = \pm j\sqrt{3}$. We can conclude that $T_{\max} = 1/3$ and by (12) the maximum time-delay is $\tau_{\max} = 2\pi/3\sqrt{3}$. This means that this is the maximum value that can be taken by the time-delay before the system becomes unstable. Now if we consider polynomial parametric uncertainty involved in the parameters of the model we have

$$s + 1q_1q_2^2 + 2\left(\frac{1-Ts}{1+Ts}\right)^2 = 0. \tag{25}$$

From (23) we have

$$\begin{aligned} p(s,q,T) &= T^2s^3 + \left(2T + T^2q_1q_2^2 + 2T^2\right)s^2 \\ &\quad + \left(2Tq_1q_2^2 - 4T + 1\right)s + 2 + q_1q_2^2, \end{aligned} \tag{26}$$

where the Hurwitz matrix $H[p(s,q,T)]$ is

$$\begin{bmatrix} 2T + T^2q_1q_2^2 + 2T^2 & 2 + q_1q_2^2 & 0 \\ T^2 & 2Tq_1q_2^2 - 4T + 1 & 0 \\ 0 & 2T + T^2q_1q_2^2 + 2T^2 & 2 + q_1q_2^2 \end{bmatrix}. \tag{27}$$

We applied the definitions presented before to analyze the positivity of the leading principal minors. Running the algorithm, we took the values of $q_i \in [0,1]$ and $T \in [0, 0.2]$; Figure 2 was obtained.

We can see in Figure 2 that the $(\alpha_i^{\text{inf}}, \beta_i^{\text{inf}})$ points symbolized with "+" and $(\alpha_i^{\text{der}}, \beta_i^{\text{der}})$, $(\alpha_i^{izq}, \beta_i^{izq})$ symbolized with "$*$" are located above the α axis, where according to *Robust Stability Analysis Algorithm*, the system is robustly stable.

If we analyze a small variation in the time-delay with values of $T \in [0, 0.4]$, we have Figure 3. We can see that the points $(\alpha_i^{\text{inf}}, \beta_i^{\text{inf}})$ symbolized with "+" and $(\alpha_i^{\text{der}}, \beta_i^{\text{der}})$, $(\alpha_i^{izq}, \beta_i^{izq})$ symbolized with "$*$" take negative values below the axis α, which—in relation to the *Robust Stability Analysis Algorithm*—means that the conditions that guarantee the robust stability of the system are not satisfied.

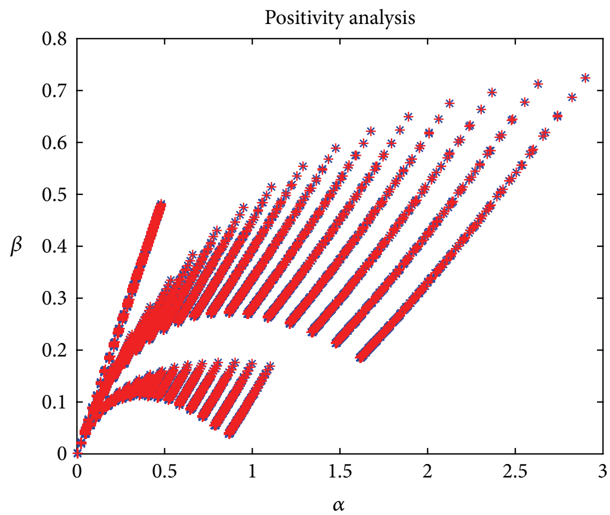

Figure 2: Sign definite decomposition of the Hurwitz matrix for the example.

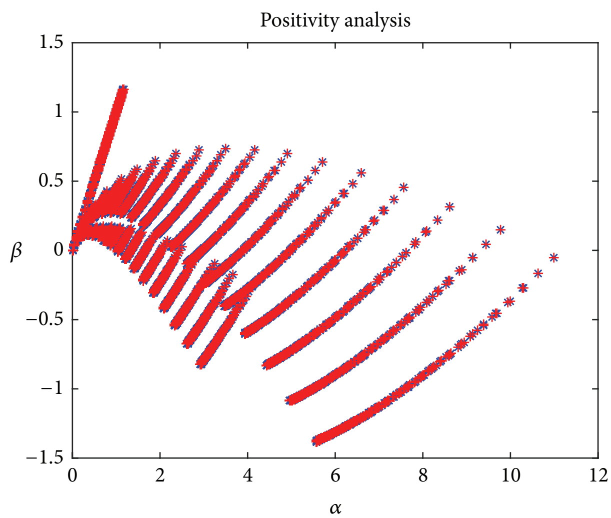

Figure 3: Sign definite decomposition of the determinant of the Hurwitz matrix for the example with a little variation in the time-delay.

7. Conclusions

In this research it was shown that the robust stability property of linear dynamical systems, which have polynomic uncertain parameters and time-delay, can be verified by the application of an algorithm based on the method of sign definite decomposition. The positivity of the determinant of the Hurwitz matrix is verified by checking the positivity of all leading principal minors of the matrix in terms of (α, β) representation. This Hurwitz matrix contains the elements of the polynomial from the transformation of the characteristic quasi-polynomial in an auxiliary equation in terms of T. For future research, the next step is to optimize the algorithm to perform faster computation time and new methods to identify the positivity of the determinants of the Hurwitz matrix.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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